Lesson 45

Introduction to Coordinate Proofs

A coordinate proof is a style of proof that uses coordinate geometry and algebra. In a coordinate proof, a diagram is used that is placed on the coordinate plane. Figures can be placed anywhere on the plane, but it is usually easiest to place one side on an axis or to place one vertex at the origin.

Example 1 Positioning a Figure on the Coordinate Plane

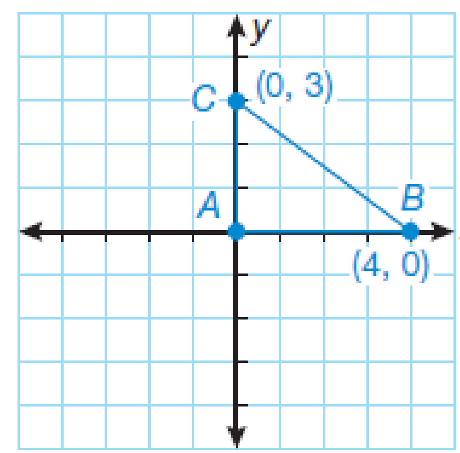
Triangle *ABC* has a base of 4 units and a height of 3 units. Angle *A* is a right angle. Position ΔABC on the coordinate plane.

SOLUTION

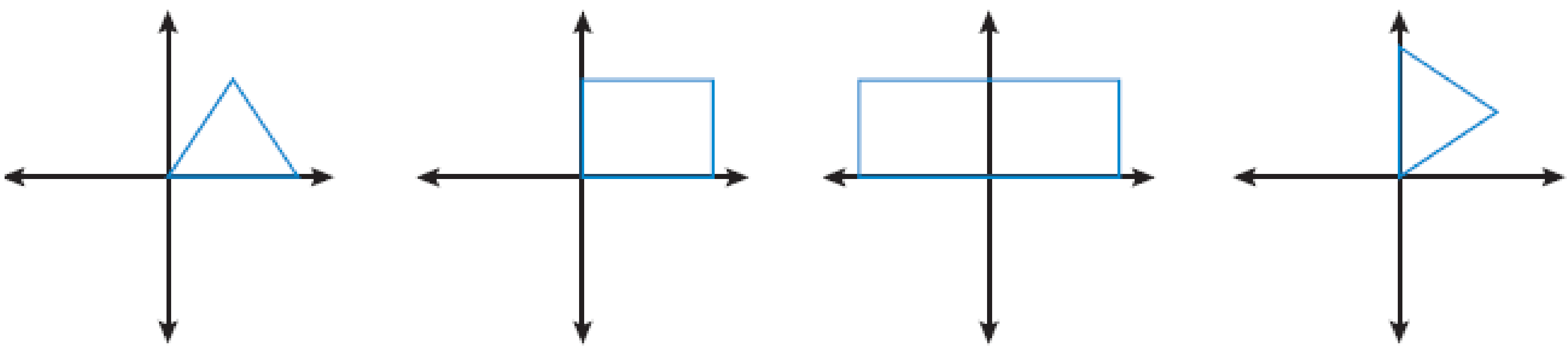
There are various ways to position the triangle on the coordinate plane. A simple way is to use the origin (0, 0) as the vertex for *A*.

Place one of the legs of the triangle on the *x*-axis, and place the other leg on the *y*-axis.

On the *x*-axis, label *B*(4, 0). On the *y*-axis, label *C*(0, 3). Draw the triangle.



When a figure is placed in a convenient position on the coordinate plane, the equations and values used in a proof will be easier to work with. Below are examples of convenient placement for common figures.



Example 2 Writing a Proof Using Coordinate Geometry

Use a coordinate proof to show that ΔHIJ is an isosceles triangle.

SOLUTION

IfΔHIJ is isosceles then, by definition, two of its sides must have equal length. Calculate each of the side lengths to verify that ΔHIJ is an isosceles triangle.

$$JI = \sqrt{\left(x_j - x_i\right)^2 + \left(y_j - y_i\right)^2} \qquad HJ = \sqrt{\left(x_j - x_h\right)^2 + \left(y_j - y_h\right)^2} \qquad HI = \sqrt{(x_h - x_i)^2 + (y_h - y_i)^2}$$

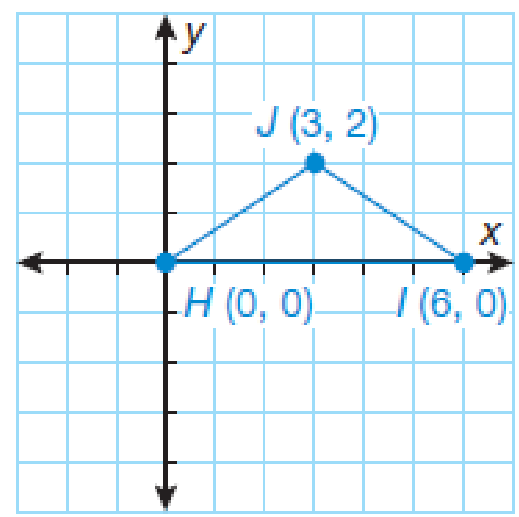
$$JI = \sqrt{(3-6)^2 + (2-0)^2} \qquad HJ = \sqrt{(3-0)^2 + (2-0)^2} \qquad HI = \sqrt{(6-0)^2 + (0-0)^2}$$

$$JI = \sqrt{(-3)^2 + (2)^2} \qquad HJ = \sqrt{(3)^2 + (2)^2} \qquad HI = \sqrt{(6-0)^2 + (0-0)^2}$$

$$JI = \sqrt{9+4} \qquad HJ = \sqrt{9+4} \qquad HI = \sqrt{6^2}$$

$$JI = \sqrt{13} \qquad HJ = \sqrt{13} \qquad HI = 6$$

Since $\overline{JI}$ and $\overline{HJ}$ are the same length, ΔHIJ is an isosceles triangle.



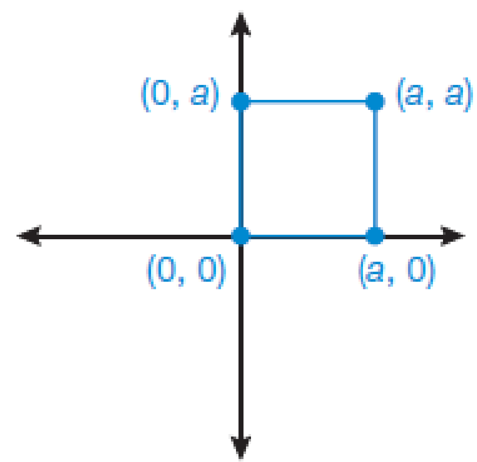
Sometimes a figure's dimensions might be unknown. When placing a figure with unknown dimensions on the coordinate plane, pick a convenient position and label the vertices of the figure using information that is given in the problem.

Example 3 Assigning Variable Coordinates to Vertices

a. A square has a side length, *a*. Place the square on the coordinate plane and label each vertex with an ordered pair.

SOLUTION

Place one vertex at the origin. Label the vertex at the origin (0,0). Because the vertex on the *x*–axis is *a* units away from the origin, its coordinates should be labeled (*a*, 0). The vertex on the *y*–axis is *a* units up from the origin, so its coordinates are (0, *a*). Finally, the fourth vertex is both *a* units to the right of the origin and *a* units up from the origin, at (*a*, *a*).



Example 3 Assigning Variable Coordinates to Vertices

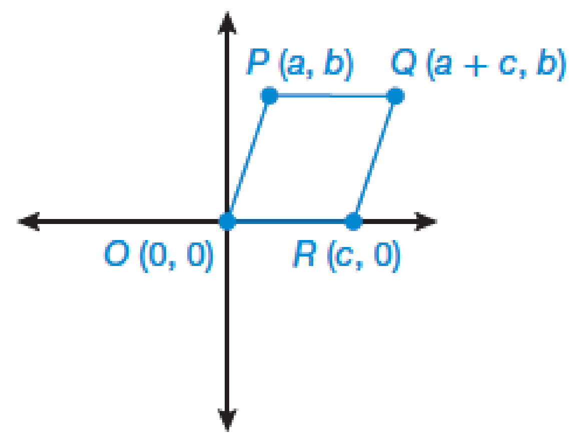
b. Given the parallelogram *OPQR*, with one side length labeled *c*, assign possible coordinates to the vertices.

SOLUTION

Place vertex *O* at (0, 0) and $\overline{OR}$ along the positive *x*–axis. Label vertices *P*, *Q*, and *R*.

Assign $OR = c$, so the coordinates for *R* are $(c, 0)$.

Give *P* the coordinates (a, b). Because *OPQR* is a parallelogram, $PQ = OR$. Therefore, the *x*–coordinate of *Q* is the *x*–coordinate of *P* plus *c* units, or $a + c$. The coordinates of *Q* are $(a + c, b)$.

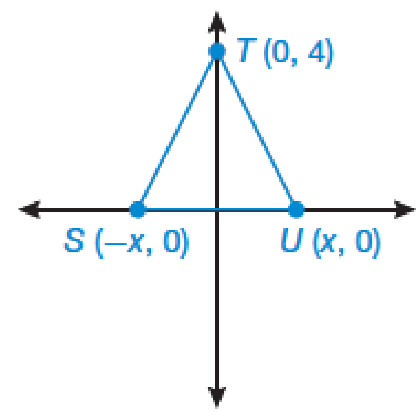


Example 3 Assigning Variable Coordinates to Vertices

c. Assign coordinates to the vertices of isosceles ΔSTU with a height of 4 from the vertex.

SOLUTION

Place vertex *T* on the *y*–axis so that its coordinates are (0, 4). If points *S* and *U* are placed such that they are equally distant from the *y*–axis, then they will form two right triangles with congruent hypotenuses. This ensures that the figure is an isosceles triangle. The coordinates of *S* and *U* are $(-x, 0)$ and $(x, 0)$, respectively.



When you assign variable coordinates to a figure used in a proof, remember that the values you choose must apply to all cases. When the dimensions of a figure are not given, variables must be used to ensure the proof is valid for a figure of any size.

Example 4 Writing a Coordinate Proof

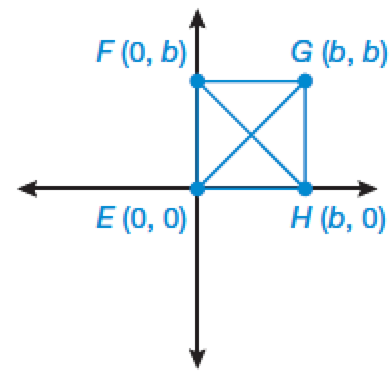
Prove that the diagonals of a square are perpendicular to one another.

SOLUTION

Assign square *EFGH* a side length of *b*. Place *E* at (0, 0), *F* at (0, *b*), *G* at (*b*, *b*), and *H* at (*b*, 0).

Draw the diagonals $\overline{FH}$ and $\overline{GE}$.

Calculate the slope of diagonals $\overline{FH}$ and $\overline{GE}$.



$m_{FH} = \frac{0-b}{b-0}$

$m_{FH} = \frac{-b}{b}$

$m_{FH} = -1$

$m_{GE} = \frac{b-0}{b-0}$

$m_{GE} = \frac{b}{b}$

$m_{GE} = 1$

Because the product of the two slopes is −1, $\overline{FH} \perp \overline{GE}$.

Notice that the slopes of the diagonals do not depend on the value of *b*. Therefore, for all squares, it is true that the diagonals of the square are perpendicular to each other.

Example 5 Application: Constructing a Swimming Pool

A contractor has been hired to build a swimming pool with a smaller wading pool beside it. The contractor draws a diagram of what he plans to build and overlays a coordinate grid on it, as shown. Show that the wading pool has a surface area that is one eighth the size of the larger pool's surface area.

SOLUTION

The area of a rectangle is lw. The pool in the diagram has a length of *c* and a width of *d*, so its total area is *cd*. The wading pool is a triangle.

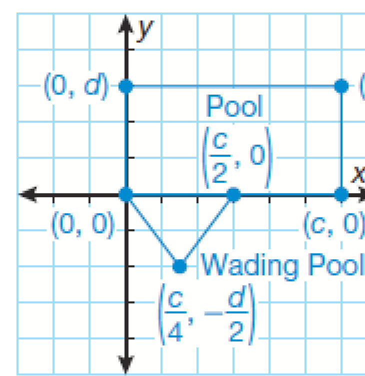
The area of a triangle is $\frac{1}{2}bh$. The height of the wading pool is $\frac{d}{2}$ and the length of its base is $\frac{c}{2}$. Substitute these values into the formula for area of a triangle.

$$A = \frac{1}{2}bh$$

$$A = \frac{1}{2}\left(\frac{c}{2}\right)\left(\frac{d}{2}\right)$$

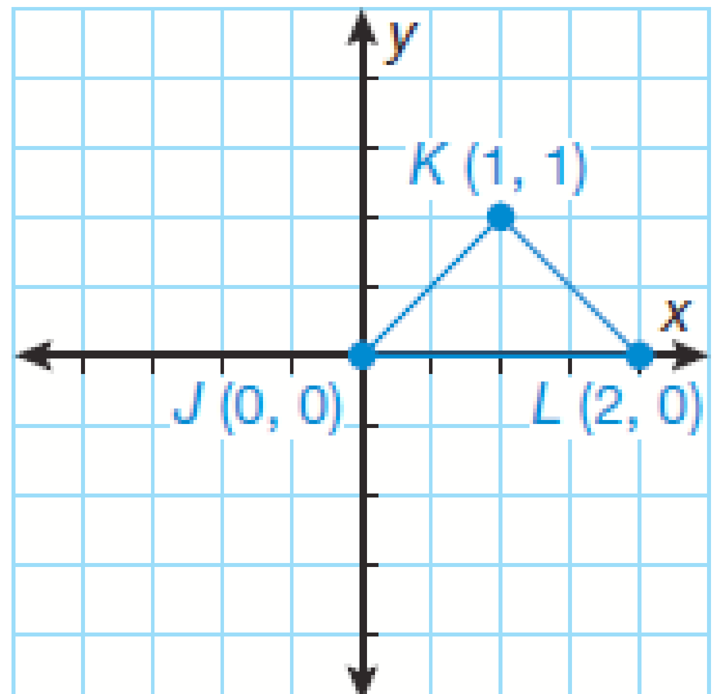
$$A = \frac{cd}{8}$$

`Therefore, the surface area of the wading pool is one-eighth the surface area of the swimming pool.



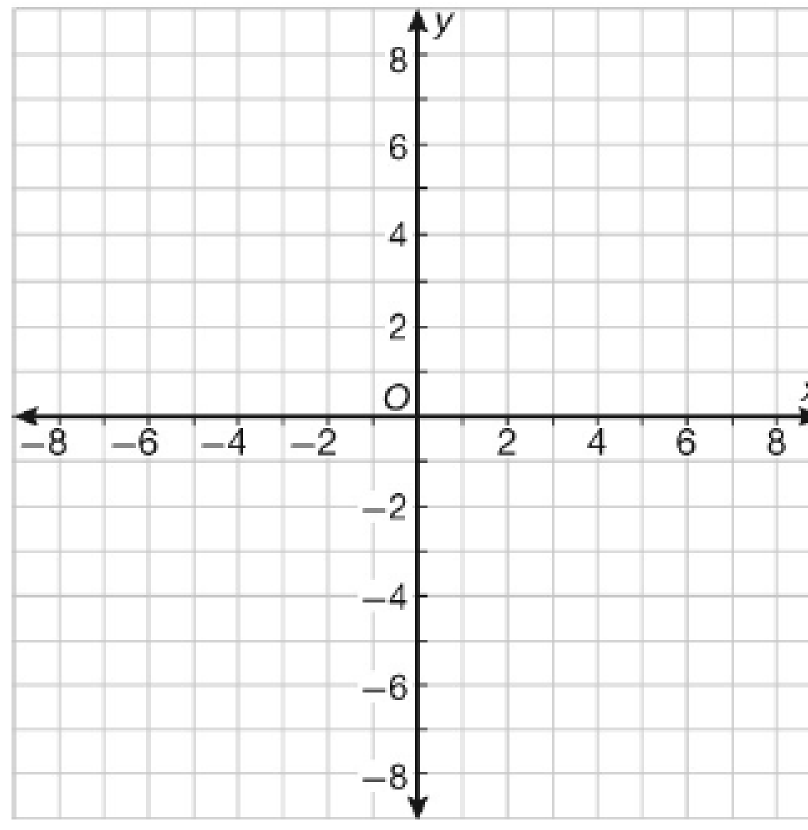
You Try!!!!

b.Prove that ΔJKL is an isosceles triangle.



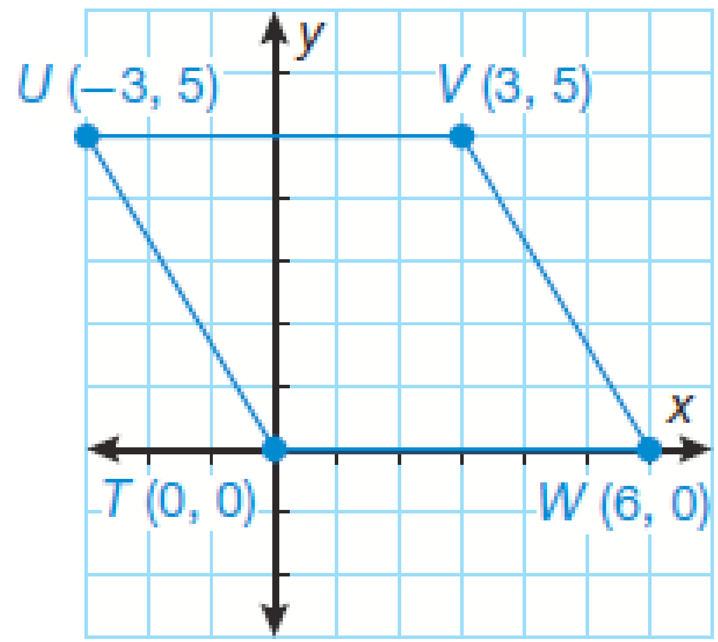
You Try!!!!

c. Place a right triangle with leg lengths of *a* and *b* units on the coordinate plane. Label the vertices with their coordinates.



You Try!!!!

d. Prove that figure *TUVW* is a parallelogram.



Assignment

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Lesson Practice (Ask Mr. Heintz)

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Practice 1–30 (Do the starred ones first)